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ON THE FREE VIBRATION OF STEPPED BEAMS

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Abstract—On the basis of a first-order shear deformation beam theory, a finite element method is presented for the analysis of free vibration of arbitrarily stepped beams. The effects of shear deformation, step geometry, step eccentricity and multiple stepped sections are investigated. The phenomenon of dynamic stiffening is also analysed and discussed. The numerical results show the importance of including bending–extension coupling for certain eccentrically stepped beams.

1. INTRODUCTION

Stepped beams are frequently found in structures due to fabrication, assembly, stiffness and space constraints. Many studies have been reported on the dynamic behavior of stepped beams. Balasubramanian and Subramanian (1985) and Subramanian and Balasubramanian (1987,1989) investigated the effect of a single stepped section on the free vibration of stepped beams. Using Euler beam theory, Jang and Bert (1989a,b) reported the exact solutions for the natural frequencies and mode shapes of a stepped beam under various boundary conditions. Laura *et al.* (1991) presented some experimental results for the natural frequencies of stepped beams. The phenomenon of dynamic stiffening created by a stepped section was also discussed in their work. Yuan and Dickinson (1992), using the Rayleigh–Ritz method with artificial torsional and linear springs connecting the stepped and the main sections, reported the natural frequencies for a stepped beam with various boundary conditions.

In the above studies, the neutral axes of the stepped section and the main section are assumed to be collinear for the whole span of the beam. Moreover, the beams are restricted to those having a single stepped section. In practical applications, stepped beams may not be created in this manner due to considerations of fabrication and assembly. Such beams are said to be eccentrically stepped as the neutral axes of the stepped section and the main section of the beam are no longer collinear. An example of such a stepped beam is a beam which has a flat bottom surface and a top surface with a step variation in profile. In this case, coupling between bending and extension is induced. Beams with multiple steps may also be desirable for various structural applications.

In the present study, a first-order shear deformation beam theory and the corresponding finite element formulation are presented. This finite element model can be used to analyse the dynamic characteristics of beams with multiple stepped sections. In the model, the effect of bending–extension coupling is included for eccentrically stepped beams. The effects of shear deformation and the geometric parameters on the natural frequencies and mode shapes are discussed. The phenomenon of dynamic stiffening is also analysed and discussed.

2. THEORY AND FORMULATION

Finite element formulation

Based on the first-order shear deformation Timoshenko beam theory, the displacement field is assumed to be of the form

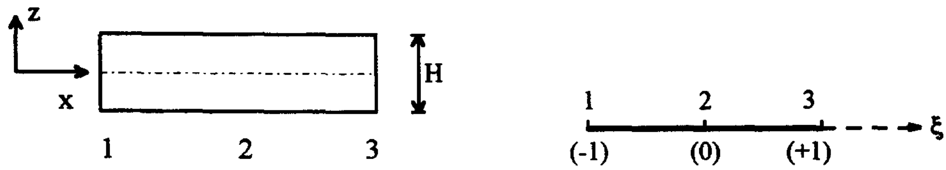

Fig. 1. A three-noded beam element and its intrinsic coordinates.

$$u(x,z) = u_0(x) + z\varphi_x(x)$$
$$w(x,z) = w_0(x) \tag{1}$$

where $u_0(x)$ and $w_0(x)$ are the axial and transverse displacements of the mid-plane, respectively, and $\varphi_x(x)$ is the rotation of the plane normal to the mid-plane.

From eqn (1), the normal strain ε_x and shear strain γ_{xz} are

$$\varepsilon_x = \frac{\partial u_0}{\partial x} + z\frac{\partial \varphi_x}{\partial x}$$
$$\gamma_{xz} = \frac{\partial w_0}{\partial x} + \varphi_x. \tag{2}$$

For a very thin beam, the shear deformation can be ignored, that is, $\gamma_{xz} = 0$, and eqn (1) reduces to the Euler beam theory.

As shown in Fig. 1, the beam element has three nodes, and each node has three degrees of freedom, namely, u_{0i}, w_{0i} and φ_{xi}, respectively. The displacements $u_0(x)$ and $w_0(x)$, and the rotation $\varphi_x(x)$ in eqn (1) can thus be interpolated in terms of the intrinsic coordinate ξ as

$$u_0 = \sum_{i=1}^{3} N_i(\xi) u_{0i}$$
$$w_0 = \sum_{i=1}^{3} N_i(\xi) w_{0i}$$
$$\varphi_x = \sum_{i=1}^{3} N_i(\xi) \varphi_{xi} \tag{3}$$

where $N_i(\xi)$, with $i = 1$–3, are the interpolation functions given by

$$N_1(\xi) = -\xi(1-\xi)/2$$
$$N_2(\xi) = (1-\xi^2)$$
$$N_3(\xi) = \xi(1+\xi)/2. \tag{4}$$

Substitution of eqns (3) and (4) into eqn (2) gives

$$\{\varepsilon\} = [B]\{\delta\} \tag{5}$$

where

$$\{\varepsilon\} = [\varepsilon_x \, \gamma_{xz}]^{\mathrm{T}}$$
$$[B] = [B_1 \, B_2 \, B_3]$$

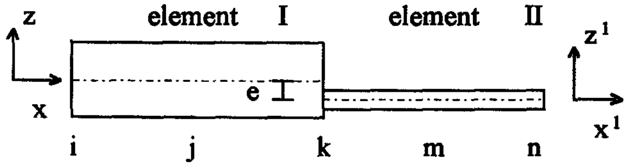

Fig. 2. Elements at a connecting node k.

$$\{\delta\} = [\delta_1\, \delta_2\, \delta_3]^{\mathrm{T}} \tag{5a}$$

and

$$[B_i] = \begin{bmatrix} N_{i,x} & 0 & zN_{i,x} \\ 0 & N_{i,x} & N_i \end{bmatrix}, \quad i = 1\text{–}3 \tag{5b}$$

$$\{\delta_i\} = [u_{0i}\, w_{0i}\, \varphi_{xi}]^{\mathrm{T}}, \quad i = 1\text{–}3. \tag{5c}$$

The stress–strain relations are

$$\begin{Bmatrix} \sigma_x \\ \tau_{xz} \end{Bmatrix} = [D] \begin{Bmatrix} \varepsilon_x \\ \gamma_{xz} \end{Bmatrix} \tag{6}$$

where

$$[D] = \begin{bmatrix} E & 0 \\ 0 & kG \end{bmatrix} \tag{6a}$$

and E and G are the Young's modulus and shear modulus of the beam. The shear correction factor k used in this study is 5/6.

The element stiffness and mass matrices are

$$[k] = \int_{l_e} \iint_{A_e} [B]^{\mathrm{T}}[D][B]\mathrm{d}_A \mathrm{d}_x$$

$$[m] = \int_{l_e} \iint_{A_e} \rho [N]^{\mathrm{T}}[N]\mathrm{d}_A \mathrm{d}_x \tag{7}$$

where l_e and A_e are the length and the cross-sectional area of the beam element, respectively, ρ is the density of the material of the element, and

$$[N] = \begin{bmatrix} N_1 & 0 & zN_1 & N_2 & 0 & zN_2 & N_3 & 0 & zN_3 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 \end{bmatrix}. \tag{7a}$$

It can be seen from eqns (7) and (7a) that the transverse, in-plane and rotatory inertia is naturally included in the element mass matrix $[m]$.

Displacement continuity conditions

Consider the two elements at the connecting node k shown in Fig. 2. The element nodal displacement vectors and the element stiffness and mass matrices of element I and element II are $\{\delta\}$, $\{\delta\}^1$, $[k]$, $[k]^1$, $[m]$ and $[m]^1$, respectively, where $\{\delta\}$ and $\{\delta\}^1$ are defined as

$$\{\delta\} = [u_{0i}, w_{0i}, \varphi_{xi}, u_{0j}, w_{0j}, \varphi_{xj}, u_{0k}, w_{0k}, \varphi_{xk}]^{T}$$

$$\{\delta\}^{1} = [u^{1}_{0k}, w^{1}_{0k}, \varphi^{1}_{xk}, u^{1}_{0m}, w^{1}_{0m}, \varphi^{1}_{xm}, u^{1}_{0n}, w^{1}_{0n}, \varphi^{1}_{xn}]^{T}. \tag{8}$$

The displacement continuity at node k is enforced by assuming

$$\begin{cases} w_{0k} = w^{1}_{0k} \\ \varphi_{xk} = \varphi^{1}_{xk} \end{cases}. \tag{9a}$$

For the axial displacement, the following equation can be established

$$u_{0k} - e\varphi_{xk} = u^{1}_{0k}. \tag{9b}$$

Here, the eccentricity e of two beam sections with a step is considered positive when the neutral axis of the left section is at a higher level than the right one.

Using the above relations, the element nodal displacement vector of element II can be expressed as

$$\{\delta\}^{1} = [T]\{\bar{\delta}\}^{1} \tag{10}$$

where the displacement vector $\{\bar{\delta}\}^{1}$ and the transformation matrix $[T]$ are

$$\{\bar{\delta}\}^{1} = [u_{0k}, w_{0k}, \varphi_{xk}, u^{1}_{0m}, w^{1}_{0m}, \varphi^{1}_{xm}, u^{1}_{0n}, w^{1}_{0n}, \varphi^{1}_{xn}]^{T} \tag{10a}$$

and

$$[T] = \mathrm{Diag}[A, I, I] \tag{10b}$$

with

$$[A] = \begin{bmatrix} 1 & 0 & -e \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad [I] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{10c}$$

Consequently, the element stiffness and mass matrices for element II are transformed as:

$$[\bar{k}]^{1} = [T]^{T}[k]^{1}[T]$$

$$[\bar{m}]^{1} = [T]^{T}[m]^{1}[T]. \tag{11}$$

The matrices $[\bar{k}]^{1}$ and $[\bar{m}]^{1}$ are used to assemble the global stiffness matrix and mass matrix for element II.

Calculation of modal parameters

The following eigenvalue equation is obtained by assembling all the element matrices:

$$([K] - \omega^{2}[M])\{\Delta\} = \{0\} \tag{12}$$

where $[K]$ and $[M]$ are the global stiffness and mass matrices, respectively, ω is the natural frequency and $\{\Delta\}$ is the corresponding mode shape.

The subspace iteration method (Bathe and Wilson, 1976) is used in the finite element program to solve the above eigenvalue problem.

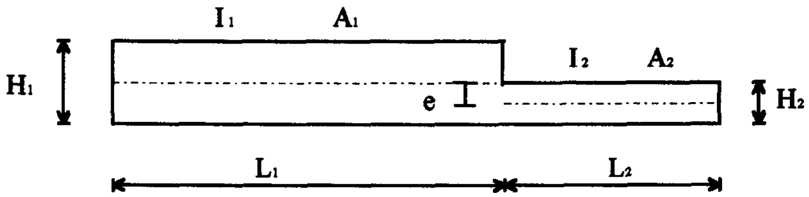

Fig. 3. A typical stepped beam.

3. NUMERICAL RESULTS AND DISCUSSION

An eccentrically stepped beam, shown in Fig. 3, is investigated in order to study the dynamic behavior of stepped beams. The cross-section of the stepped beam can be rectangular or circular. The parameters $H_{(1,2)}$, $L_{(1,2)}$, $I_{(1,2)}$ and $A_{(1,2)}$ are the height, length, area moment of inertia and the cross-sectional areas of the main section and the remaining section of the beam.

For convenience, the following geometric parameters are defined:

$$\begin{aligned} L &= L_1 + L_2 \\ \gamma_L &= L_2/L \\ \gamma_H &= H_2/H_1 \\ \gamma_I &= I_2/I_1 \\ \gamma_A &= A_2/A_1 \\ \gamma_{HL} &= H_1/L. \end{aligned} \tag{13}$$

The convergence of the present method and the corresponding finite element program is checked by varying the number of elements for the beam. For the stepped beam, convergence to at least five significant figures is obtained when the mesh is between 10 and 30. It is important to point out that if the ratio of the height to length of the beam is very small, shear locking may occur for the present model just as for other C^0 continuous plate and shell elements. In this case, the reduced and selective integration methods and other methods suggested by Huang (1989), Subramanian and Balasubramanian (1989) and Ramesh Babu *et al.* (1987) could be adopted to overcome this problem.

Comparison with exact solutions

In order to validate the present model and the corresponding finite element program, the natural frequencies of stepped beams with circular cross-sections under clamped–clamped and clamped–free boundary conditions having various γ_I are computed. Results are reported for the zero eccentricity case in Tables 1 and 2, along with exact solutions (presented within parenthesis) by Jang and Bert (1989b). It is found that the relative differences between the present results and the exact solutions are less than 2%. Although the reported exact solutions were obtained by using the Euler theory, the effect of shear deformation can be expected to be very small because of the small ratio of height to length (1/25) used in the computations. Also, as expected, the present results are always smaller than the corresponding exact solutions due to the effect of shear deformation and rotatory inertia. In conclusion, the accuracy of the present model is sufficient for engineering applications.

Effect of shear deformation

The effect of shear deformation on the natural frequencies of stepped beams is dependent on the beam geometry, material properties and boundary conditions. Figures 4a and 4b show this effect due to the variation in the length of the stepped section for stepped beams of clamped–clamped and clamped–free boundary conditions. The cross-section of

Table 1. Frequency coefficient β_i of a circular cross-section clamped–free stepped beam: $\gamma_A = \sqrt{\gamma_I}$, $\gamma_L = 1{:}2$, $\gamma_{HL} = 1/25$, $e = 0$

$$\omega_i = \beta_i \sqrt{\frac{EI_1}{\rho A_1 L^4}}$$

γ_I	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5
1	3.5143	21.775	61.432	120.59	199.40
	(3.5160)	(22.234)	(61.697)	(120.902)	(199.860)
5	2.4351	22.1250	78.194	141.98	245.20
	(2.4373)	(22.0345)	(78.559)	(142.572)	(245.589)
10	2.0413	20.984	85.232	154.88	258.87
	(2.0629)	(21.0943)	(85.625)	(155.515)	(259.312)
20	1.7343	19.274	89.735	174.26	266.65
	(1.7418)	(19.3670)	(90.143)	(174.940)	(267.045)
40	1.4614	17.302	91.718	199.64	273.07
	(1.4685)	(17.3857)	(92.129)	(200.362)	(273.521)

Figures in parentheses are from Jang and Bert (1989b).

Table 2. Frequency coefficient β_i of a circular cross-section clamped–clamped stepped beam: $\gamma_A = \sqrt{\gamma_I}$, $\gamma_L = 1{:}2$, $\gamma_{HL} = 1/25$, $e = 0$

$$\omega_i = \beta_i \sqrt{\frac{EI_1}{\rho A_1 L^4}}$$

γ_I	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5
1	22.271	61.403	120.51	199.44	297.90
	(22.3733)	(61.6742)	(120.903)	(199.859)	(298.556)
5	25.831	77.788	141.50	245.09	358.56
	(25.9531)	(78.1518)	(142.088)	(245.592)	(359.097)
10	27.542	84.974	153.87	258.83	398.64
	(27.6807)	(85.3656)	(254.595)	(259.252)	(398.97)
20	30.175	89.802	172.56	266.41	443.67
	(30.3213)	(90.2097)	(173.279)	(266.839)	(444.351)
40	34.157	92.135	197.56	272.44	473.79
	(34.3252)	(92.5507)	(198.276)	(272.912)	(474.506)

Figures in parentheses are from Jang and Bert (1989b).

the stepped beam is rectangular. For the clamped–free beam, the thinner section is at the free end. The percentage difference in natural frequencies, Δf, is defined as:

$$\Delta f = \frac{f_E - f_S}{f_S} \times 100\% \tag{14}$$

where f_E and f_S are the frequencies obtained without and with shear deformation. The results for f_E are obtained independently using an Euler beam element.

The present results show that the effect of shear deformation on the higher modes is more significant than on the lower modes, and more significant for clamped–clamped beams than for clamped–free beams. As expected, it can been seen that the shorter the length of the stepped section, the stronger the effect.

Effect of eccentricity of the stepped section

If a beam is eccentrically stepped, that is, $e \neq 0$ in Fig. 3, the coupling of bending and extension is induced because the neutral axes of bending of the stepped section and main section are not collinear. In this case, the bending–extension coupling must be included in the vibration analysis. A stepped cantilever beam and a stepped clamped–clamped beam of rectangular cross-section are considered in order to investigate this kind of effect. As in the previous case, the percentage difference in natural frequencies can be defined as:

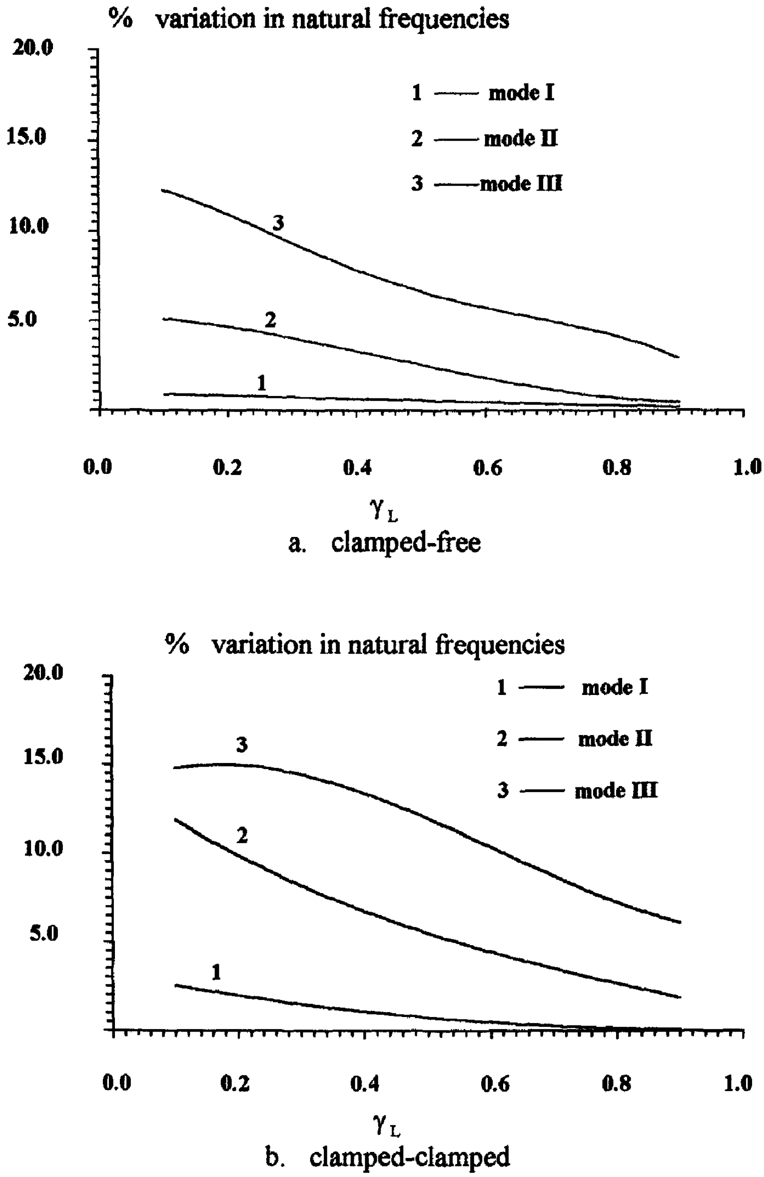

Fig. 4. Effect of shear deformation on natural frequencies of stepped beams.

$$\Delta f = \frac{f_{e \neq 0} - f_{e=0}}{f_{e \neq 0}} \times 100\% \tag{15}$$

where $f_{e \neq 0}$ is the frequency for the case of $e \neq 0$, while $f_{e=0}$ is the frequency for the case of $e = 0$.

The results presented in Table 3 show that the effect of the eccentricity of the stepped section on the natural frequencies depends on the boundary conditions of the stepped beam. For a clamped–free beam with the thinner section at the free end, this kind of effect can be negligible. However, for a clamped–clamped beam, the effect is noticeable. As expected, due to the increase in stiffness caused by the bending–extension coupling in eccentrically stepped beams, the natural frequencies are higher than the natural frequencies of beams with non-eccentrically stepped sections.

Effect of thickness variation across a step

The effect of thickness variation across a step is examined by keeping H_1 constant while varying H_2. The ratios of the natural frequencies of the stepped beam with respect to the natural frequencies of a uniform beam of thickness H_1 are presented in Figs 5 and 6.

Table 3. Percentage difference (%) in natural frequencies due to eccentricity of the stepped section: $\gamma_A = 1/3$, $\gamma_I = 1/27$, $\gamma_H = 1/3$, $\gamma_{HL} = 1/15$, $e = H_1/3$

Boundary	Clamped–free			Clamped–clamped		
γ_L	Mode 1	Mode 2	Mode 3	Mode 1	Mode 2	Mode 3
0.1	0.4	1.9	1.1	2.2	8.0	3.5
0.2	0.4	0.2	0.9	5.3	6.1	5.6
0.4	0.5	1.6	0.8	7.0	2.7	2.4
0.6	0.1	0.3	0.7	2.1	2.1	1.9
0.8	0.3	0.2	0.6	4.3	1.5	2.0

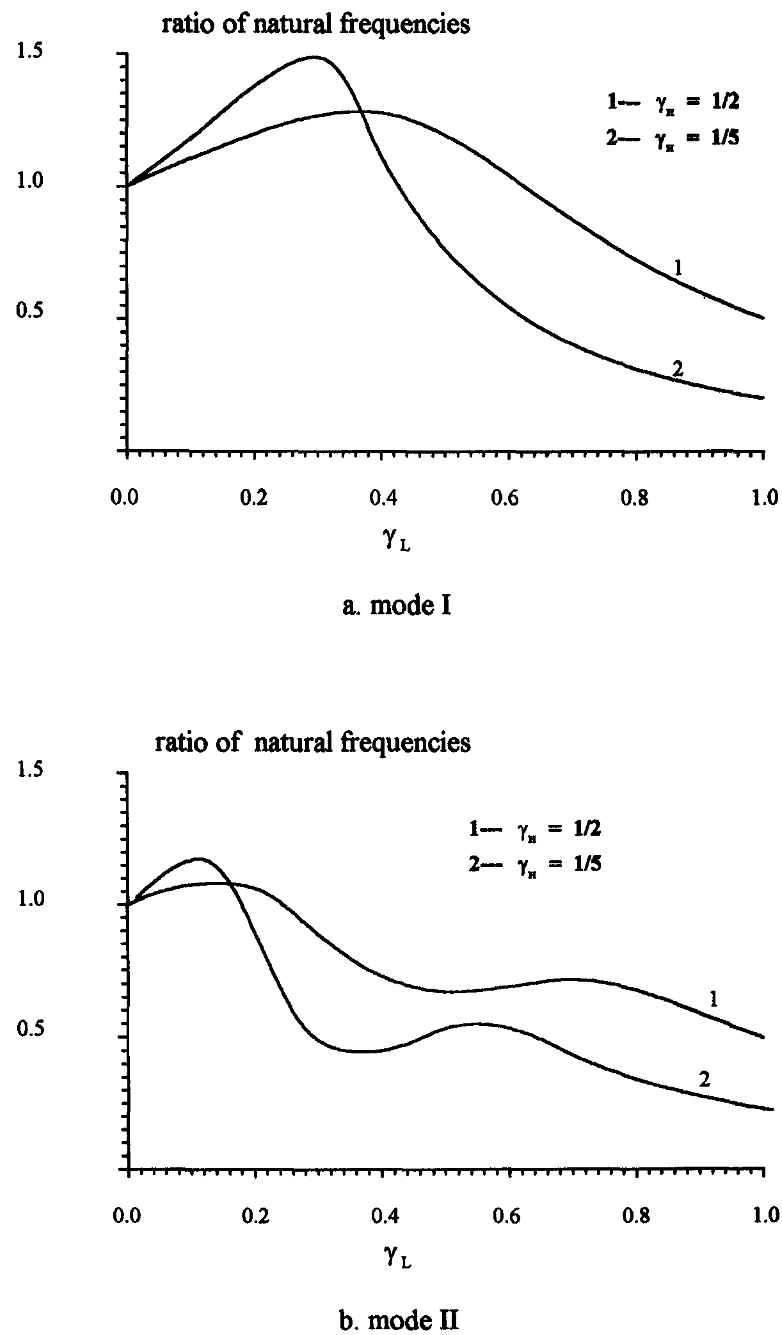

Fig. 5. Effect of thickness variation across a step on the natural frequencies of a stepped beam with clamped–free boundary conditions.

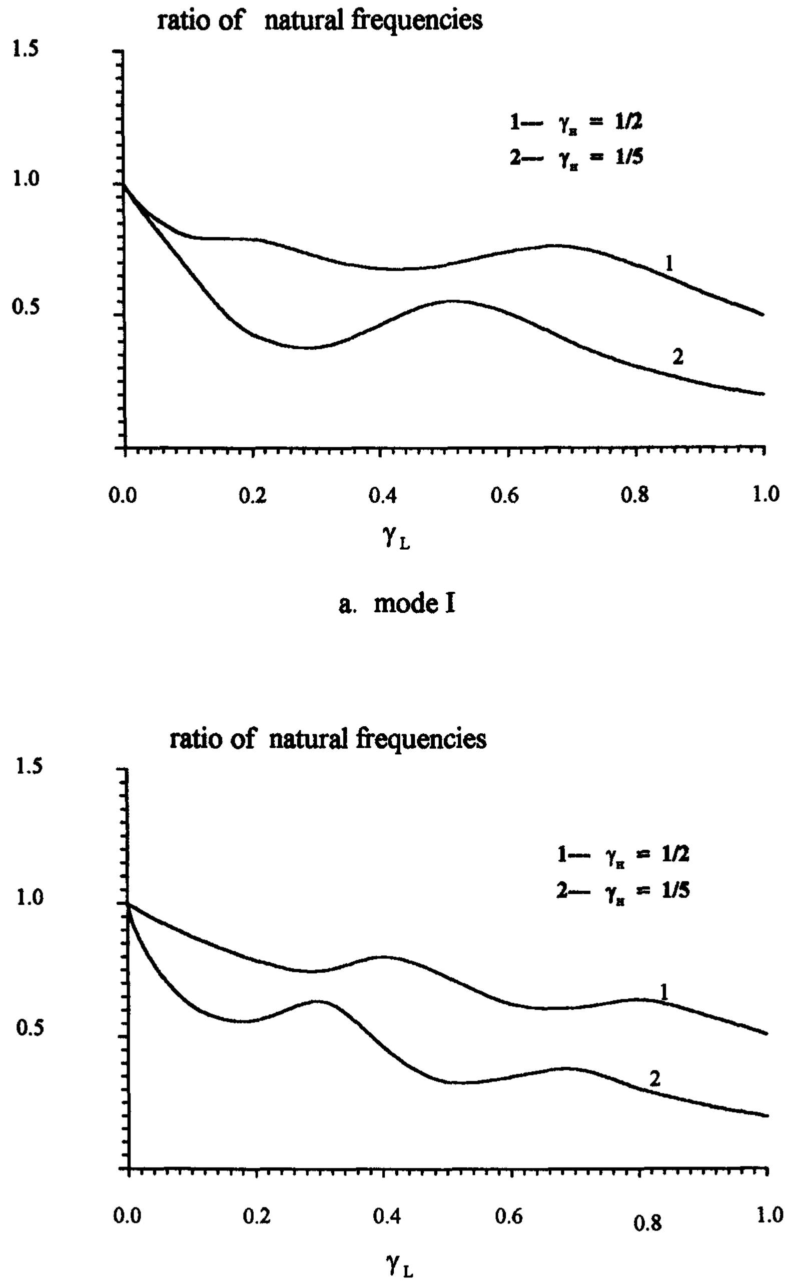

Fig. 6. Effect of thickness variation across a step on the natural frequencies of a stepped beam with clamped–clamped boundary conditions.

It can be seen that the variation of natural frequencies of a stepped beam compared to the corresponding uniform beam is dependent on the beam geometry and the boundary conditions. For a clamped–free beam with the thinner section at the free end, and for lower value of γ_L, the increase in natural frequencies is significant. This increase in natural frequency can be of the order of 50% for the first mode. It is interesting that this kind of increase in frequencies is obtained by reducing the cross-section, or, in other words, by removing material from a uniform beam. This phenomenon is called "dynamic stiffening", and has some practical applications (Subramanian and Balasubramanian, 1987; Laura,

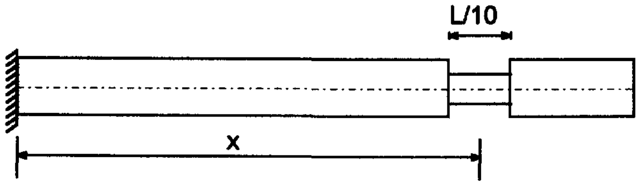

Fig. 7. A beam with a stepped section of $L/10$ length.

1991). The reason for the many peaks and troughs in Figs 5 and 6 is that the natural frequencies of stepped beams are dependent on both the relative loss of stiffness and mass caused by the presence of the stepped section. This effect is discussed in detail in the following section.

On the "dynamic stiffening" of stepped beams

Geometrically, a stepped beam is created by removing material from a section of a corresponding uniform beam. This leads to the loss of both mass and stiffness. Generally, a reduction in stiffness ΔI can cause a decrease in natural frequency, and a reduction in mass Δm can cause an increase in natural frequency. Compared to a uniform beam, the natural frequency of a stepped beam is dependent not only on reductions in mass and stiffness but also on the location where the stepped section is created. Figure 7 shows a stepped beam where the location x/L of the step section varies, and the length of the step section is 1/10 of the beam length.

To investigate the individual effects of Δm and ΔI for a clamped–free beam, first, let $\Delta I = 0$ and $\Delta m \neq 0$ (by artificially modifying the matrices in the finite element program). The ratios of natural frequencies of the present stepped beam to a corresponding uniform beam are computed and are presented as curve 1 in Fig. 8 as a function of the location of the stepped sections; second, let $\Delta m = 0$ and $\Delta I \neq 0$, curve 2 is obtained; third, let $\Delta m \neq 0$ and $\Delta I \neq 0$, and curve 3 is computed.

It can be seen from Fig. 8 that the amount of increase (due to the reduction in mass Δm) and the amount of decrease (due to the reduction in stiffness ΔI) are quite different, and the final natural frequency depends on both the above factors although the effect of ΔI is generally dominant. When the stepped section is located near the free end, the natural frequency of a stepped beam may be higher than that of the corresponding uniform beam.

Effect of multiple stepped sections

Figure 9 shows a uniform beam (beam I) and two beams with multiple stepped sections (beam II and beam III). Beam III is eccentrically stepped with a flat surface. Table 4 shows the natural frequency coefficients of the three beams for clamped–free and clamped–clamped boundary conditions.

The results show that for the clamped–free boundary conditions, the difference between the natural frequencies of beam II and those of beam III is slight. However, for the clamped–clamped boundary condition, the difference is noticeable for the fundamental frequency. In the case of the clamped–free boundary condition, the fundamental frequencies of both beam II and beam III are about 16% higher than the fundamental frequency of the corresponding uniform beam (beam I). For the clamped–clamped beam with multiple stepped sections, the natural frequencies show a significant decrease compared to the natural frequencies of the corresponding uniform beam.

4. CLOSURE

On the basis of a first-order shear deformation theory, a finite element model is presented for the analysis of free vibration of stepped beams. The model can be used to

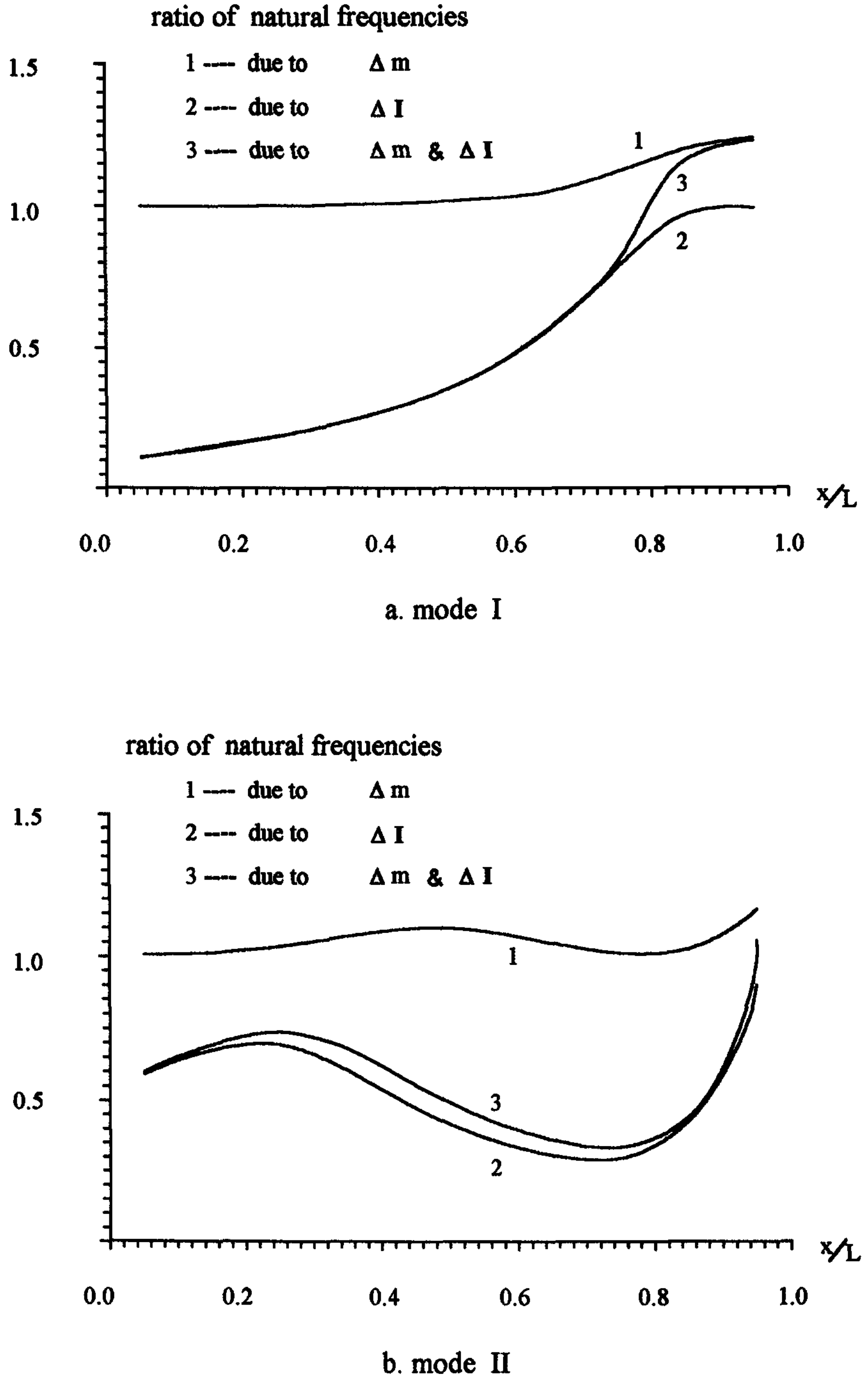

Fig. 8. Effects of Δm, ΔI and both Δm and ΔI on the natural frequencies of a stepped beam with clamped–free boundary conditions.

compute the natural frequencies and mode shapes of beams with multiple eccentrically stepped sections. The numerical results show good agreement with available exact solutions. The effect of shear deformation, beam geometry, bending–extension coupling of eccentrically stepped beams and multiple stepped sections on the natural frequencies and mode shapes are presented for beams with clamped–clamped and clamped–free boundary conditions. The dynamic stiffening phenomenon is also discussed.

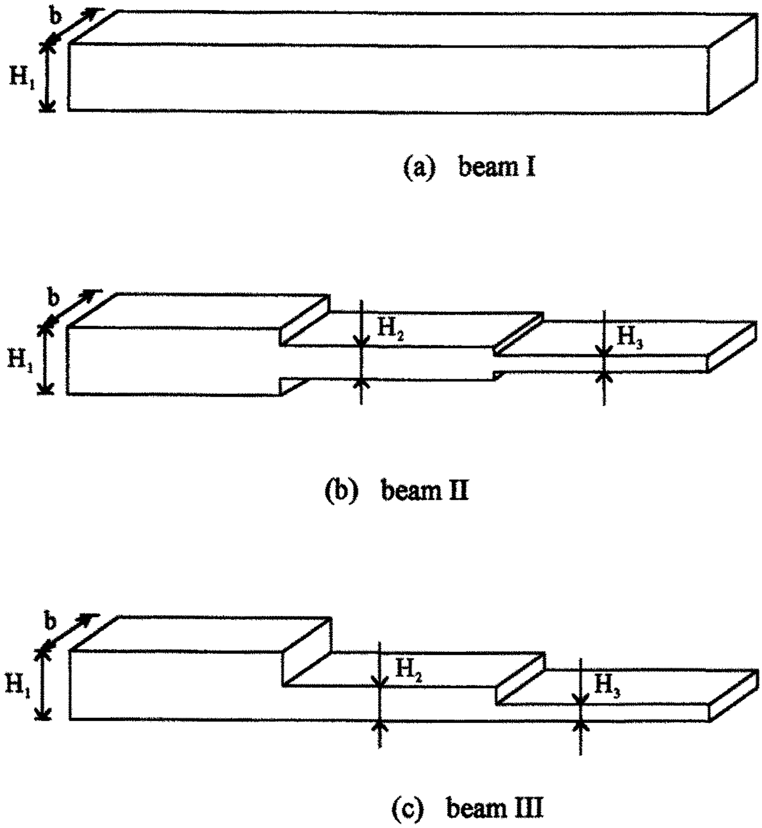

Fig. 9. A uniform beam and two beams with multiple stepped sections.

Table 4. Natural frequency coefficients β_i of a uniform beam and beams with multiple steps:

$$\omega_i = \beta_i \sqrt{\frac{EI_1}{\rho A_1 L^4}}$$

Boundary condition	Beam pattern	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5
C–F	Beam I	3.5107	21.738	61.407	120.41	199.38
	Beam II	4.1191	12.318	28.934	55.158	93.571
	Beam III	4.1201	12.324	28.934	55.163	93.640
C–C	Beam I	22.264	61.383	120.42	199.39	297.79
	Beam II	11.680	28.969	55.222	93.627	143.50
	Beam III	12.525	29.174	55.437	93.750	144.12

C–F clamped–free; C–C clamped–clamped.

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